Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Linear systems: iterative methods

Note: in this series you will need the MATLAB functions implemented in Series 9.

Exercise I (MATLAB)

Consider the general linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$.

a) Write a MATLAB function `preconditioned_gradient.m` that implements the preconditioned gradient method for the solution of the linear system. The stopping criterion should be based on the norm of the residual: $r^{(k)} = \left\|\mathbf{r}^{(k)}\right\| = \left\|\mathbf{b} - A\mathbf{x}^{(k)}\right\| < tol$. Use the function `preconditioned_gradient_template.m` as template:

```
function [ x, k, res ] = preconditioned_gradient( A, b, P, x0, tol, kmax )
% PRECONDITIONED_GRADIENT solve the linear system A x = b by means
% of the Preconditioned Gadrient method; the preconditioning matrix must be
% non singular. Stopping criterion based on the residual.
%  [ x, k, res ] = preconditioned_gradient( A, b, P, x0, tol, kmax )
%  Inputs:  A    = matrix (square matrix)
%           b    = vector (right hand side of the linear system)
%           P    = preconditioning matrix (non singular, same size of A)
%           x0   = initial solution (colum vector)
%           tol  = tolerence for the stopping driterion based on residual
%           kmax = maximum number of iterations
%  Outputs: x    = solution vector (column vector)
%           k    = number of iterations at convergence
%           res  = value of the norm of the residual at convergence
%

return
```

(*Hint*: use the MATLAB command \ to solve, at each step of the iterative method, the linear system involving the preconditioning matrix P).

b) Set $n = 4$, $A = \begin{bmatrix} 5 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 0 & 1 & 1 & 5 \end{bmatrix}$, which is symmetric and positive definite, and $\mathbf{b}$ such that the exact solution is $\mathbf{x} = (1, \ldots, 1)^T$. Solve the linear system $A\mathbf{x} = \mathbf{b}$ by means of `preconditioned_gradient`, without preconditioning, i.e. setting $P = P_1 = I \in \mathbb{R}^{n \times n}$, the identity matrix. Then, set as preconditioner $P = P_2 \in \mathbb{R}^{n \times n}$ the tridiagonal matrix extracted from A. Set `tol`$= 10^{-6}$ and $\mathbf{x}^{(0)} = (0, \ldots, 0)^T$.

c) Consider the Jacobi and Gauss-Seidel methods. When applied to solve the system from point b), do they converge for any choice of $\mathbf{x}^{(0)}$?

d) Plot the norms of the residuals $r^{(k)}$ vs. the number of iterations k for the preconditioned gradient methods with $P = P_1$ and P_2 given at point b), but also those obtained with the Jacobi and Gauss-Seidel methods. Which of the methods converges faster? (*Hint*: set `tol`$= 10^{-14}$ and calculate the norms of the residuals corresponding to the iterative methods for `kmax`$= 1, \ldots, 50$.).

Exercise II (MATLAB)

Consider the general linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$.

a) Define A as the Hilbert matrix of dimension $n \times n$, i.e. the symmetric and positive definite matrix such that $(A)_{ij} = \frac{1}{i+j-1}$, $i, j = 1, \ldots, n$. Specifically, set $n = 15$. Is the stopping criterion based on the *relative* residual satisfactory for the solution of a linear system associated to the matrix A by means of an iterative method?

b) Verify the answer given at point a) for the Gauss-Seidel method. Define $\mathbf{b}$ so that the exact solution is $\mathbf{x} = (1, \ldots, 1)^T$. Set $\mathbf{x}^{(0)} = (0, \ldots, 0)^T$. Consider the stopping criterion based on the relative residual (instead of the criterion on the norm of the residual implemented in Series 9), with a tolerance $tol_{rel} = 10^{-5}$.

c) Consider the tridiagonal matrix

$$A = \begin{bmatrix} (4+\beta) & -2 & 0 & & & & \\ -2 & (4+\beta) & -2 & 0 & & & \\ 0 & -2 & (4+\beta) & -2 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & 0 & -2 & (4+\beta) & -2 \\ & & & & 0 & -2 & (4+\beta) \end{bmatrix}, \quad \text{with } \beta > 0,$$

and let $\beta = 5 \cdot 10^{-5}$. When solving a linear system with matrix A by means of the Gauss-Seidel method, is the stopping criterion based on the difference between successive iterates satisfactory?

d) Verify the answer given at point c) by solving the linear system with $\mathbf{b}$ corresponding to the exact solution $\mathbf{x} = (1, \ldots, 1)^T$. Set `tol`$= 10^{-5}$ and $\mathbf{x}^{(0)} = (0, \ldots, 0)^T$. Note that the MATLAB function `gauss_seidel.m` from Series 9 should be modified to implement the stopping criterion based on the difference of successive iterates.

Exercise III (Theoretical)

Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x},\mathbf{b} \in \mathbb{R}^n$. For its numerical solution, we use an iterative method, where $\mathbf{x}$ is obtained as the limit of the sequence $\left\{\mathbf{x}^{(k)}\right\}_{k=0}^{\infty}$. The iterative method is in the form:

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{g}, \qquad k = 0, 1, \ldots,$$

for a given initial solution $\mathbf{x}^{(0)}$, iteration matrix $B \in \mathbb{R}^{n\times n}$, and vector $\mathbf{g} \in \mathbb{R}^n$. We assume that there exists an invertible preconditioner $P \in \mathbb{R}^{n\times n}$ such that $B = I - P^{-1}A$ and $\mathbf{g} = P^{-1}\mathbf{b}$.

Specifically, we set $n = 2$ and $A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$.

a) Set $P = P_1 = \beta D$, where $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix extracted from A and $\beta > 0$ is a parameter. Which iterative method do we obtain for $\beta = 1$? Calculate the values of β for which this iterative method converges for all $\mathbf{x}^{(0)}$.

b) Calculate the value of β for which the iterative method from point a) exhibits the fastest convergence rate (for any choice of $\mathbf{x}^{(0)}$).

c) Repeat point a) by setting $P = P_2 = \beta\,(D - E)$, where E is the lower triangular matrix such that $(E)_{ij} = -(A)_{ij}$ for $i = 2, \ldots, n$, $j = 1, \ldots, i-1$.

d) Repeat point b) by considering the preconditioning matrix P_2 from point c).

e) Which of the two preconditioning matrices P_1 or P_2 and which value of β would you choose to ensure the fastest convergence of the iterative method, starting from a generic initial solution $\mathbf{x}^{(0)}$?